To Infinity and Beyond

Rebekah Yates

Houghton College, Houghton, NY

Finite Sets

Do these sets have the same size?

- $\{1, 2, 3, 4, 5\}$ and $\{2, 3, 4, 5, 6\}$

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- $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $\{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

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- set of Snow White's dwarves and
  set of players on the Ultimate frisbee field for one team

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  $\{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

- set of Snow White's dwarves and
  set of players on the Ultimate frisbee field for one team

- set of tennis ball cans
  set of tennis balls that go in those cans

Conclusion

The only way two finite sets can have the same size is if their elements can be put in a one-to-one correspondence.

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Definition

A nonempty set S is **finite** *if there is a natural number n such that there is a one-to-one correspondence between the set S and the set* $\{1, 2, \ldots, n\}$.

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Definition

A nonempty set S is **finite** *if there is a natural number n such that there is a one-to-one correspondence between the set S and the set $\{1, 2, \ldots, n\}$.*

Implication: an ordered finite set has a largest element.

Definition of Infinite

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Example

- the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- the set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- the set of rational numbers $\mathbb{Q}$
- the set of real numbers $\mathbb{R}$
- the set of all possible strings of letters

Natural Numbers

Proposition

The set of all natural numbers is infinite.

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The set of all natural numbers is infinite.

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Assume that the ordered set of natural numbers $\mathbb{N}$ is finite.
Then there is a largest natural number; we'll call that number n.
But $n + 1 > n$ and $n + 1$ is a natural number.

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Natural Numbers

Proposition

The set of all natural numbers is infinite.

Proof by Contradiction.

Assume that the ordered set of natural numbers $\mathbb{N}$ is finite.
Then there is a largest natural number; we'll call that number n.
But $n + 1 > n$ and $n + 1$ is a natural number.
This is a contradiction! Hence, $\mathbb{N}$ must be infinite. $\square$

Definition of Cardinality

Definition

Two sets A and B have the same **cardinality** *if there is a one-to-one correspondence between the elements of A and the elements of B.*

Examples

Example

Compare the set $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ with the set $\mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, 6, \ldots\}$.

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Compare the set $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ with the set $\mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, 6, \ldots\}$.

Possible one-to-one correspondence:

$$\begin{gathered} 1 \leftrightarrow 2 \\ 2 \leftrightarrow 3 \\ 3 \leftrightarrow 4 \\ \vdots \end{gathered}$$

Examples

Example

Compare the set of even natural numbers $2\mathbb{N} = \{2, 4, 6, 8, 10, \ldots\}$ with $\mathbb{N}$.

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Example

Compare the set of even natural numbers $2\mathbb{N} = \{2, 4, 6, 8, 10, \ldots\}$ with $\mathbb{N}$.

Possible one-to-one correspondence:

$$\begin{aligned} 1 &\leftrightarrow 2 \\ 2 &\leftrightarrow 4 \\ 3 &\leftrightarrow 6 \\ &\vdots \end{aligned}$$

Examples

Example

Compare the set of reciprocals of natural numbers with $\mathbb{N}$.

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Compare the set of reciprocals of natural numbers with $\mathbb{N}$.

Possible one-to-one correspondence:

$$\begin{aligned} 1 &\leftrightarrow 1 \\ 2 &\leftrightarrow \frac{1}{2} \\ 3 &\leftrightarrow \frac{1}{3} \\ &\vdots \end{aligned}$$

Examples

Example

Compare the integers $\mathbb{Z}$ with $\mathbb{N}$.

Possible one-to-one correspondence:

$$\begin{aligned} 1 &\leftrightarrow 0 \\ 2 &\leftrightarrow 1 \\ 3 &\leftrightarrow -1 \\ 4 &\leftrightarrow 2 \\ 5 &\leftrightarrow -2 \\ 6 &\leftrightarrow 3 \\ 7 &\leftrightarrow -3 \\ &\vdots \end{aligned}$$

Example

Compare the rational numbers $\mathbb{Q}$ with $\mathbb{N}$

Consider this diagram

$$\begin{array}{cccccccccccc}
 & & & & & & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & & & & & & 3 & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
 & & & & & & 2 & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\
 & & & & & & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
 & & & & & 0 & & & & & \\
\cdots & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 & & & & & & \\
\cdots & -\frac{2}{4} & -\frac{2}{3} & -\frac{2}{2} & -2 & & & & & & \\
\cdots & -\frac{3}{4} & -\frac{3}{3} & -\frac{3}{2} & -3 & & & & & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & & & & & &
\end{array}$$

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Diagram Continued

$$
\begin{array}{ccccccccccc}
 & & & & & & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & & & & & & 3 & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
 & & & & & & 2 & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\
 & & & & & & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
 & & & & & \boxed{0} & & & & & \\
\cdots & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 & & & & & & \\
\cdots & -\frac{2}{4} & -\frac{2}{3} & -\frac{2}{2} & -2 & & & & & & \\
\cdots & -\frac{3}{4} & -\frac{3}{3} & -\frac{3}{2} & -3 & & & & & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & & & & & &
\end{array}
$$

Diagram Continued

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
3 & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
2 & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots
\end{array}
$$

$$
\boxed{0} \rightarrow 1
$$

$$
\begin{array}{ccccc}
\cdots & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 \\
\cdots & -\frac{2}{4} & -\frac{2}{3} & -\frac{2}{2} & -2 \\
\cdots & -\frac{3}{4} & -\frac{3}{3} & -\frac{3}{2} & -3 \\
\iddots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Diagram Continued

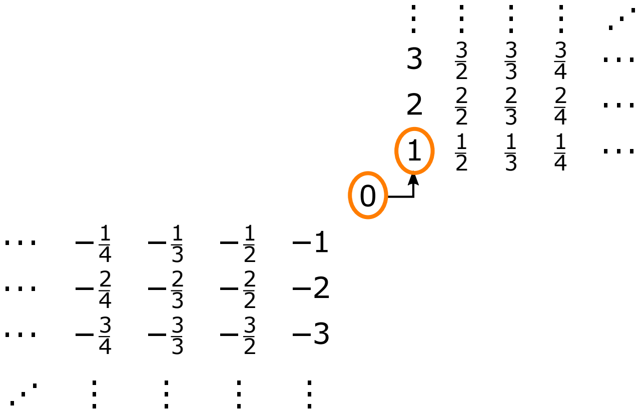
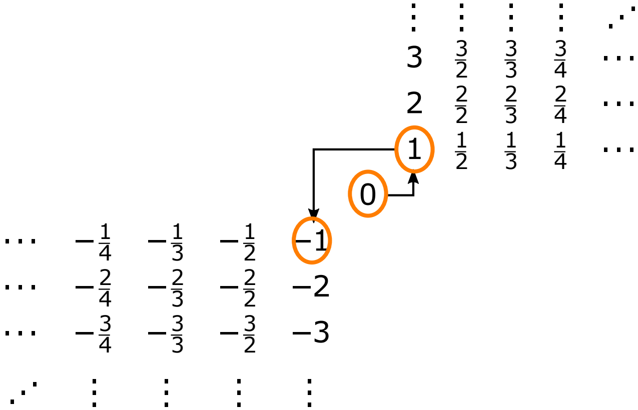


Diagram Continued



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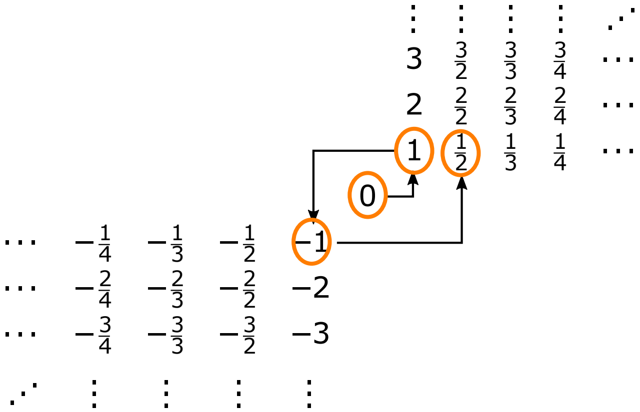

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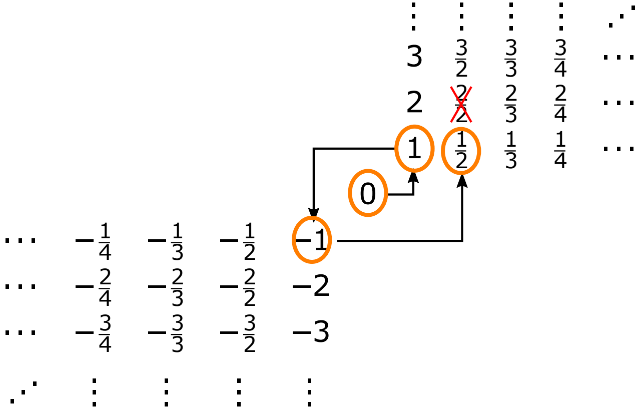

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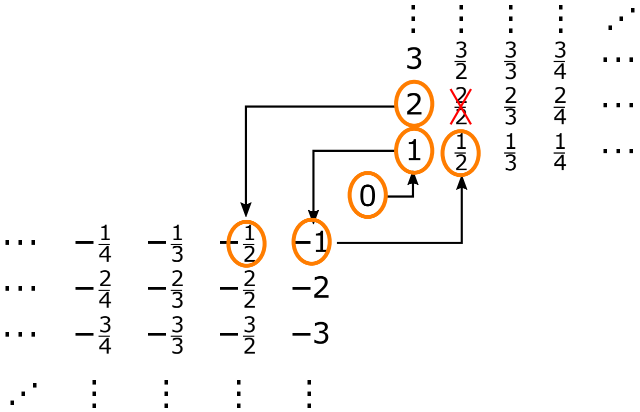

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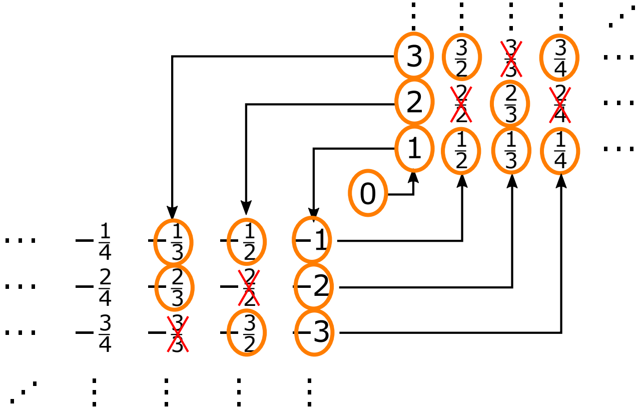

Question

Are there any infinite sets that *don't* have the same cardinality as $\mathbb{N}$?

Real Numbers

Theorem

The real numbers are uncountable.

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Proof. Assume that the real numbers in the interval $(0, 1)$ are countable.

Real Numbers

Theorem

The real numbers are uncountable.

Proof. Assume that the real numbers in the interval $(0,1)$ are countable. Then we can make a list $\{r_1, r_2, r_3, r_4, r_5, r_6, \ldots\}$ which includes all such numbers:

$$\begin{aligned}
r_1 &= 0.r_{11}r_{12}r_{13}r_{14}r_{15}r_{16}\ldots \\
r_2 &= 0.r_{21}r_{22}r_{23}r_{24}r_{25}r_{26}\ldots \\
r_3 &= 0.r_{31}r_{32}r_{33}r_{34}r_{35}r_{36}\ldots \\
r_4 &= 0.r_{41}r_{42}r_{43}r_{44}r_{45}r_{46}\ldots \\
r_5 &= 0.r_{51}r_{52}r_{53}r_{54}r_{55}r_{56}\ldots \\
r_6 &= 0.r_{61}r_{62}r_{63}r_{64}r_{65}r_{66}\ldots \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots
\end{aligned}$$

Proof, continued

Now construct a real number $s = 0.s_1 s_2 s_3 s_4 \ldots$ as follows:

$$s_n = \begin{cases} 1 & \text{if } r_{nn} \neq 1 \\ 2 & \text{if } r_{nn} = 1. \end{cases}$$

Example

$0.\mathbf{1}23456789\ldots$

$0.2\mathbf{1}3451235\ldots$

$0.87\mathbf{2}347623\ldots$

$0.234\mathbf{1}34323\ldots$

$0.7259\mathbf{8}1234\ldots$

$0.67589\mathbf{4}462\ldots$

$0.987654\mathbf{5}67\ldots$

$0.7658653\mathbf{7}3\ldots$

$0.87345729\mathbf{8}\ldots$

Proof, continued

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Example

$0.\mathbf{1}23456789\ldots$	$0.\mathbf{2}$
$0.2\mathbf{1}3451235\ldots$	
$0.87\mathbf{2}347623\ldots$	
$0.234\mathbf{1}34323\ldots$	
$0.7259\mathbf{8}1234\ldots$	
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ON

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$0.87\mathbf{2}347623\ldots$	$0.22\mathbf{1}$
$0.234\mathbf{1}34323\ldots$	
$0.7259\mathbf{8}1234\ldots$	
$0.67589\mathbf{4}462\ldots$	
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0.**1**23456789 . . .	0.**2**
0.2**1**3451235 . . .	0.2**2**
0.87**2**347623 . . .	0.22**1**
0.234**1**34323 . . .	0.221**2**
0.7259**8**1234 . . .	
0.67589**4**462 . . .	
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0.7658653**7**3 . . .	
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$0.87\mathbf{2}347623\ldots$	$0.22\mathbf{1}$
$0.234\mathbf{1}34323\ldots$	$0.221\mathbf{2}$
$0.7259\mathbf{8}1234\ldots$	$0.2212\mathbf{1}$
$0.67589\mathbf{4}462\ldots$	$0.22121\mathbf{1}$
$0.987654\mathbf{5}67\ldots$	$0.221211\mathbf{1}$
$0.7658653\mathbf{7}3\ldots$	$0.2212111\mathbf{1}$
$0.87345729\mathbf{8}\ldots$	

ON

Proof, continued

Now construct a real number $s = 0.s_1 s_2 s_3 s_4 \ldots$ as follows:

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0.67589**4**462 . . .	0.22121**1**
0.987654**5**67 . . .	0.221211**1**
0.7658653**7**3 . . .	0.2212111**1**
0.87345729**8** . . .	0.22121111**1**

Proof, continued

Then s is not in the original list $\{r_1, r_2, r_3, \ldots\}$ since it differs by at least one digit from each of the numbers in that list.
Hence, the original list is not an enumeration of the real numbers in the interval $(0, 1)$.
We have reached a contradiction, so our assumption that the real numbers are countable is false. $\square$

The Mathematician Behind Infinity

Cantor vs. Infinite Inertia

"The infinite is recognizable but not comprehensible." (Descartes 1596–1650)

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"In mathematics infinite magnitude may never be used as something final; infinity is only a *facon de parler* [manner of speaking], meaning a limit to which certain ratios may approach as closely as desired when others are permitted to increase indefinitely." (Gauss in 1831)

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"...I realise that in this undertaking I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers." (Cantor in 1883)

More Opposition

"Later generations will regard set theory as a disease from which one has recovered." (Poincaré in 1908)

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Kronecker called Cantor a "scientific charlatan," a "renegade" and a "corrupter of youth."

Approval

"The solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our age has to boast." (Russell)

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"This appears to me to be the most admirable flower of the mathematical intellect and one of the highest achievements of purely rational human activity." (Hilbert)

G. H. Hardy

"[The mathematician's] subject is the most curious of all—there is none in which truth plays such odd pranks."

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"A mathematician, like a painter or poet, is a maker of patterns.... The mathematician's patterns, like the painter's or the poet's must be *beautiful*; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics...."