Part I (due Friday, January 19 at the beginning of class)

Here's a reminder of what reduced row echelon form is. A matrix in *row echelon* form has the first three characteristics but not necessarily the last characteristic on this list. (Quiz note: I will never ask you to write down a definition for reduced row echelon form or for row echelon form, but I may ask you to identify if a particular matrix is in one of these forms or not and be able to explain why it is not if the answer is no or to give an example and explain it.)

Reduced Row Echelon Form (rref)

- The first nonzero entry in row 1 is a 1. *Note: we call it a* ***leading one***
- Rows of all zeros are at the bottom of the matrix.
- Each successive leading 1 is further right than the leading 1 in the row above.
- Each column with a leading 1 has zeros everywhere else in that column.

One of the reasons we really like reduced row echelon form is that rref is *unique*—each matrix has exactly one rref. Row echelon form is not unique—different sequences of row operations can lead to different row echelon forms.

Example 1. Solve the system of equations corresponding to the following augmented matrices:

(a) $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

Solution: Since this matrix is conveniently in reduced row echelon form, we can read the solution directly from the matrix; we have $x_1 = 7$, $x_2 = -12$, and $x_3 = 5$. We can write this as $(7, -12, 5)$.

(b) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$

Solution: This matrix is also conveniently already in reduced row echelon form, but we have a free variable this time. The free variable is x_3 since the third column does not have a leading one. Thus, we let $x_3 = t$ and solve for the remaining two variables in terms of t. Doing this, you should get $(3 - 2t, 4 - 2t, t)$ where $t \in \mathbb{R}$.

(c) $\begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: For this system, we have x_2 is a free variable, so we let $x_2 = t$ and solve for the remaining variables in terms of t to get the solution $(-12 - 2t, t, 3)$ where $t \in \mathbb{R}$.

(d) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Solution: Since the last line in the matrix corresponds to the equation $0 = 4$, which is invalid, this system has no solution.

Gauss–Jordan Elimination

Here are standard steps for performing elimination on an augmented matrix.

1. Locate the leftmost column with a nonzero entry.
2. Interchange two rows if necessary in order to make the first entry in the column from step 1 nonzero.
3. $a =$ entry now at the top of the column from step 1. Multiply the top row by $\frac{1}{a}$.
4. Add multiples of the top row to the rows below it so that all entries below the top leading 1 become zeros.
5. Cover the top row of the matrix and start with step 1 again on the submatrix that remains. Repeat steps 1–5 until the entire matrix is in row echelon form.
6. Starting with the last row and working upward, add multiples of each row to the rows above to get zeros above the leading 1s. Once you have done so for all leading ones, your matrix is in reduced row echelon form.

The allowable row operations are called *elementary row operations*: interchange two rows, multiply a row by a scalar, or add a multiple of one row to another row.

Try Gauss–Jordan elimination on this example and then use your calculator or Wolfram Alpha `https://www.wolframalpha.com/` (or some other computational device) to check your answer.

Example 2. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 6 & 2 & 2 & -4 \\ -4 & -5 & -2 & 12 \end{bmatrix}$

When we perform row operations on a matrix, we're turning the matrix into another form that's related to the original matrix. We call two such matrices row equivalent:

Definition 1. *A matrix A is* row equivalent *to a matrix B if A can be obtained by performing elementary row operations on B.*

Example 3. If we start with the matrix

$$A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & -4 \\ 4 & 0 & 1 \end{bmatrix}$$

and perform the row operation $2R_1 + R_2$, we get the matrix

$$B = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 0 & 10 \\ 4 & 0 & 1 \end{bmatrix},$$

which is row equivalent to A.

Often, systems of linear equations will have every equation equal to 0. We've seen this in the case of the vector equation we get when we're considering linear independence and dependence. It's a special enough case that we give it a name:

Definition 2. *A* homogeneous system *is a system of linear equations in which all the constant terms are zero.*

Here's an example of a homogeneous system:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

If we plug in 0 for every variable, we get $0 = 0$ for each of these equations, so $(0, 0, 0)$ is a solution for this homogeneous system. In fact, for any homogeneous system, $(0, 0, \ldots, 0)$ is a solution; we call it the *trivial* solution. The question then is when do we have more than just the trivial solution for a homogenous system of equations?

Theorem 1. *If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.*

A short explanation (but not a proof): each nonzero row of the matrix in rref corresponds to a leading/pivot/basic variable (since each nonzero row would have a 1 in it), so there are r leading variables, or r columns with a leading 1, leaving $n - r$ columns with no leading 1, each of which corresponds to a free variable.

Theorem 2. *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.*

Again, a short explanation: suppose the system has n unknowns and m equations, with $n > m$. If we have r nonzero rows in the rref of the augmented matrix for the system, then $r \leq m < n$, so by the previous theorem, this system must have $n - r > 0$ free variables. Thus, the system has infinitely many solutions.

Reading Question(s)

1. If the matrix in Example 2 is an augmented matrix related to a system of equations, what is the solution to that system?
2. In Example 3, is matrix B row equivalent to matrix A? Why or why not?
3. What, if any, connections do you see between the two theorems on homogeneous systems and linear dependence/independence? You do not have to be incredibly precise about these connections (or even very sure), just note anything that might be a connection.

Note: you can ask questions about things we've done in class as part of your part (b) for Part I as well as about the reading.

Part II (prepare for Friday, January 19)

I changed my mind; we're going to not do WeBWorK quite yet. For these two problems, use the standard Gauss–Jordan elimination algorithm on the augmented matrix to solve the system of equations.

1. Solve the system of equations

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

2. Solve the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 + x_5 &= 0 \\ x_2 \qquad - x_4 \qquad &= 0 \\ x_3 \qquad + x_5 &= 0 \end{aligned}$$

Part III: Homework (due Wednesday, January 24 at the beginning of class)

1. Suppose $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$. For what values of h is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent? Explain your answer.
2. True or false? If true, carefully explain your reasoning; if false, give an explained, specific counterexample.
   (a) If every variable is basic (is a leading variable), then the linear system has exactly one solution.
   (b) The presence of a free variable indicates there are no solutions to the linear system
   (c) If a linear system has exactly one solution, then it must have the same number of equations as unknowns.
   (d) If a linear system has the same number of equations as unknowns, then it has exactly one solution.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors

- span of a set of vectors
- linearly independent
- linearly dependent
- reduced row echelon form