Part I (due Monday, April 29 at the beginning of class)

Carefully read the material below, taking notes for yourself. The reading questions are below the material.

Here are some more useful properties of norms and distances in an inner product space:

Theorem 1. *If $\vec{u}$ and $\vec{v}$ are vectors in a real inner product space V, and if $k \in \mathbb{R}$, then*

(a) $\|\vec{v}\| \geq 0$, *and* $\|\vec{v}\| = 0$ *iff* $\vec{v} = \vec{0}$

(b) $\|k\vec{v}\| = |k|\,\|\vec{v}\|$

(c) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

(d) $d(\vec{u}, \vec{v}) \geq 0$, *and* $d(\vec{u}, \vec{v}) = 0$ *if and only if* $\vec{u} = \vec{v}$.

You saw the definition of orthogonal on the Inner Product Spaces handout. We often find it useful to consider all the vectors that are orthogonal to a set of vectors:

Definition 1. *If W is a subspace of an inner product space V, then the set of vectors in V that are orthogonal to every vector in W is called the* orthogonal complement *of V and denoted*

$$W^{\perp} = \{\vec{v} \in V\colon\ \langle \vec{v}, \vec{w} \rangle = 0 \text{ for all } \vec{w} \in W\}.$$

Note: $W^{\perp}$ is often read as "W perp."

Proposition 2. *If W is a subspace of an inner product space V, then $W^{\perp}$ is a subspace of V.*

Proof. Let W be a subspace of an inner product space V. Let $\vec{w} \in W$. Since $\left\langle \vec{0}, \vec{w} \right\rangle = 0$ (see theorem from handout in class), we have that $\vec{0}$ is orthogonal to every vector in W. Thus, $\vec{0} \in W^{\perp}$, so $W^{\perp} \neq \emptyset$.

Now suppose $\vec{v}_1, \vec{v}_2 \in W^{\perp}$, so $\langle \vec{v}_1, \vec{w} \rangle = 0$ and $\langle \vec{v}_2, \vec{w} \rangle = 0$. Then we have

$$\langle \vec{v}_1 + \vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + \langle \vec{v}_2, \vec{w} \rangle = 0 + 0 = 0,$$

so $\vec{v}_1 + \vec{v}_2 \in W^{\perp}$. Also, if k is a scalar, then

$$\langle k\vec{v}_1, \vec{w} \rangle = k \langle \vec{v}_1, \vec{w} \rangle = k \cdot 0 = 0,$$

so $k\vec{v}_1 \in W^{\perp}$. Hence, $W^{\perp}$ is a subspace of V. □

Proposition 3. *If A is an $m \times n$ matrix, then $\left(C(A^T)\right)^{\perp} = N(A)$ under the usual dot product (Euclidean inner product) on $\mathbb{R}^n$.*

Proof. Suppose A is an $m \times n$ matrix and let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of A. Let $\vec{v}$ be in the row space of A. Then there are scalars $k_1, k_2, \ldots, k_m$ such that

$$\vec{v} = k_1\vec{r}_1 + k_2\vec{r}_2 + \cdots + k_m\vec{r}_m.$$

Suppose $\vec{w} \in N(A)$. Then $A\vec{w} = \vec{0}$, and since we can write the matrix multiplication $A\vec{w}$ as

$$A\vec{w} = \begin{bmatrix} \vec{r}_1 \cdot \vec{w} \\ \vec{r}_2 \cdot \vec{w} \\ \vdots \\ \vec{r}_m \cdot \vec{w} \end{bmatrix},$$

we must have $\vec{r}_j \cdot \vec{w} = 0$ for all $j = 1, 2, \ldots m$, i.e., $\vec{w}$ is orthogonal to every row of A. By the properties of dot (inner) products, we have

$$\begin{aligned} \vec{v} \cdot \vec{w} &= (k_1\vec{r}_1 + k_2\vec{r}_2 + \cdots + k_m\vec{r}_m) \cdot \vec{w} \\ &= k_1\vec{r}_1 \cdot \vec{w} + k_2\vec{r}_2 \cdot \vec{w} + \cdots + k_m\vec{r}_m \cdot \vec{w} \\ &= k_1(0) + k_2(0) + \cdots + k_m(0) \\ &= 0 + 0 + \cdots + 0 = 0. \end{aligned}$$

Thus, $\vec{w} \in \left(C(A^T)\right)^{\perp}$, so $N(A) \subseteq \left(C(A^T)\right)^{\perp}$.

On the other hand, suppose $\vec{u} \in \left(C(A^T)\right)^{\perp}$. Then $\vec{u}$ is orthogonal to every vector in $C(A^T)$, i.e., $\vec{u} \cdot (k_1\vec{r}_1 + k_2\vec{r}_2 + \cdots + k_m\vec{r}_m) = 0$. By the same calculations as above but in reverse, this gives us that $\vec{u} \cdot \vec{r}_j = 0$ for all $j = 1, 2, \ldots, m$, so $\vec{u}$ is orthogonal to every row of A. Thinking about matrix multiplication as dot products of the rows of A with $\vec{u}$ again, this gives us that $A\vec{u} = \vec{0}$, so $\vec{u} \in N(A)$. Hence, $\left(C(A^T)\right)^{\perp} \subseteq N(A)$.

Therefore, $N(A) = \left(C(A^T)\right)^{\perp}$. □

This theorem allows us to easily find a basis for an orthogonal complement of a subspace in $\mathbb{R}^n$.

Suppose $W = \text{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a subspace of $\mathbb{R}^n$.

To find a basis for $W^{\perp}$: put $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ into the rows of a matrix A. Then $W = C(A^T)$ (the row space of A), so if we find a basis for $N(A)$, which is the orthogonal complement of $C(A^T)$, we'll have a basis for $W^{\perp}$.

Example 1. Find a basis for the orthogonal complement of the subspace of $\mathbb{R}^n$ spanned by the vectors $\vec{v}_1 = (3, 2, 1, 7)$, $\vec{v}_2 = (4, 1, -3, -2)$, and $\vec{v}_3 = (10, 5, -1, 12)$.

Solution: We have

$$\begin{bmatrix} 3 & 2 & 1 & 7 \\ 4 & 1 & -3 & -2 \\ 10 & 5 & -1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{7}{5} & -\frac{11}{5} \\ 0 & 1 & \frac{13}{5} & \frac{34}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for $N(A)$ is $\{(\frac{7}{5}, -\frac{13}{5}, 1, 0), (\frac{11}{5}, -\frac{34}{5}, 0, 1)\}$.

Here's one of the nice properties of orthogonal sets:

Theorem 4. *If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.*

Proof. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be an orthogonal set of nonzero vectors in an inner product space V and consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}. \qquad (1)$$

For every $\vec{v}_i \in S$, we have

$$\begin{aligned}\left\langle \vec{0}, \vec{v}_i \right\rangle &= \langle c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \vec{v}_i \rangle \\ &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + c_2 \langle \vec{v}_2, \vec{v}_i \rangle + \cdots + c_n \langle \vec{v}_n, \vec{v}_i \rangle \\ &= c_i \langle \vec{v}_i, \vec{v}_i \rangle .\end{aligned}$$

Since $\vec{v}_i \neq \vec{0}$ for all $\vec{v}_i \in S$, $\langle \vec{v}_i, \vec{v}_i \rangle \neq 0$, so we must have $c_i = 0$ for all $i = 1, \ldots, n$. Hence, S is linearly independent. □

Reading Questions

1. If V is an inner product space, what is $V^\perp$?
2. If V is a finite-dimensional inner product space and W is a subspace of V, what is $\left(W^\perp\right)^\perp$?
3. Why do we consider the equation (1) in the proof of Theorem 4?

Part II: Exercises (due by class time Monday, April 29)

Finish the yellow Inner Products handout.

Part III: Homework (due Wednesday, May 1 by 2:30 PM)

1. True or False? If true, prove; if false, give an explained counterexample.
   (a) The inner product of two vectors cannot be a negative real number.
   (b) $\langle k\vec{u}, k\vec{v} \rangle = k^2 \langle \vec{u}, \vec{v} \rangle$.
   (c) If $\vec{v}$ and $\vec{w}$ are orthogonal, then $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors
- span of a set of vectors
- linearly independent
- linearly dependent
- reduced row echelon form
- pivot

- homogeneous system
- free variable
- row equivalent
- consistent system
- inconsistent system
- trace of a matrix
- transpose of a matrix
- inverse of a matrix
- elementary matrix
- transformation
- domain
- codomain
- range
- vector space (I will not ever ask you to define this on a quiz—the definition is way too long—but you should make sure you know what makes something a vector space)
- subspace
- basis
- finite-dimensional vector space
- dimension
- coordinate vector
- column space of A
- row space of A
- null space of A
- rank
- nullity
- linear transformation
- kernel
- range
- isomorphism
- isomorphic vector spaces
- characteristic equation

- eigenvector
- eigenvalue
- diagonalizable matrix
- orthogonal vectors
- unit vector