Part I (due Friday, April 26 at the beginning of class)

Here's the theory behind the diagonalization process we used in class Wednesday:

Theorem 1. *If A is an $n \times n$ matrix, then the following are equivalent:*

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

Proof. **(a) $\implies$ (b):** Suppose A is diagonalizable. Then there exists an invertible matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

such that $P^{-1}AP = D$, a diagonal matrix, which implies that $AP = PD$. Thus, we have

$$AP = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix}$$

Let $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ be the column vectors of P. Then the column vectors of AP are

$$\lambda_1 \vec{p}_1, \lambda_2 \vec{p}_2, \ldots, \lambda_n \vec{p}_n.$$

We can also write the column vectors of AP as

$$A\vec{p}_1, A\vec{p}_2, \ldots, A\vec{p}_n.$$

Hence, we must have that

$$A\vec{p}_1 = \lambda_1 \vec{p}_1, A\vec{p}_2 = \lambda_2 \vec{p}_2, \ldots, A\vec{p}_n = \lambda_n \vec{p}_n.$$

Since P is invertible, its columns are nonzero, so $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues for A and $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ are corresponding eigenvectors. Also since P is invertible, $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ are linearly independent (from the Purple Theorem).

(b) $\implies$ (a): Suppose that A has n linearly independent eigenvectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

The column vectors of the product AP are

$$A\vec{p}_1, A\vec{p}_2, \ldots, A\vec{p}_n.$$

Since the λ_i's are eigenvalues for A and the p_i's are corresponding eigenvectors, we also have

$$A\vec{p}_1 = \lambda_1\vec{p}_1, A\vec{p}_2 = \lambda_2\vec{p}_2, \ldots, A\vec{p}_n = \lambda_n\vec{p}_n,$$

so

$$AP = \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ P_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD.$$

Since $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ are linearly independent, P must be invertible, so $AP = PD \implies P^{-1}AP = D$. Hence, A is diagonalizable. □

So we see that an $n \times n$ matrix with n linearly independent eigenvectors is diagonalizable, and we get the following method for diagonalizing A:

1. Find n linearly independent eigenvectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ for A.
2. Form a matrix P with columns $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$.
3. The matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the diagonal entries.

You don't need to turn anything in for Part I this time, but bring your questions on this reading to class!

Part II: Exercises (due by class time Friday, April 26)

Read Definition 1 on the front of the salmon Norms and Dot Products handout and try Example 1. Also read Theorem 1 on the back, and if you're feeling ambitious, try to prove any of it you can.

Part III: Homework (due Wednesday, May 1 by 2:30 PM)

1. True or False? If true, prove; if false, give an explained counterexample.
   (a) The eigenvalues of a matrix A are the same as the eigenvalues of $\text{rref}(A)$.
   (b) If A and B are invertible matrices, then AB is similar to BA.
   (c) If A is diagonalizable, then A^T is diagonalizable.
   (d) If there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of an $n \times n$ matrix A, then A is diagonalizable.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors
- span of a set of vectors
- linearly independent
- linearly dependent
- reduced row echelon form
- pivot
- homogeneous system
- free variable
- row equivalent
- consistent system
- inconsistent system
- trace of a matrix
- transpose of a matrix
- inverse of a matrix
- elementary matrix
- transformation
- domain
- codomain
- range
- vector space (I will not ever ask you to define this on a quiz—the definition is way too long—but you should make sure you know what makes something a vector space)
- subspace
- basis
- finite-dimensional vector space
- dimension
- coordinate vector
- column space of A
- row space of A

- null space of A
- rank
- nullity
- linear transformation
- kernel
- range
- isomorphism
- isomorphic vector spaces
- characteristic equation
- eigenvector
- eigenvalue
- diagonalizable matrix