Part I (due Friday, April 12 at the beginning of class)

Think some more about what things we can add to the Purple Theorem about $T_A\colon \mathbb{R}^n \to \mathbb{R}^n$ with $T(\vec{x}) = A\vec{x}$. We'll discuss these in class.

Also, read this theorem (it's the big one we've been saying is coming!!) and proof, filling in the missing parts as your reading questions:

Theorem 1. *Every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$.*

Proof. Let V be a real n-dimensional vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V. For any $\vec{w} \in V$ there exist scalars $k_1, k_2, \ldots, k_n$ such that

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n.$$

Define the transformation $T\colon V \to \mathbb{R}^n$ by

$$T(\vec{w}) = (k_1, k_2, \ldots, k_n).$$

Claim: T is an isomorphism from V to $\mathbb{R}^n$ (so we need to show T is linear, one-to-one, and onto $\mathbb{R}^n$).

Linearity: Let $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$. Then there exist scalars $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ such that

$$\vec{u} =$$

and

$$\vec{v} =$$

so

$$T(\vec{u} + \vec{v}) =$$

and

$$T(k\vec{u}) =$$

One-to-one: Suppose $T(\vec{v}) = T(\vec{w})$ for some $\vec{v}, \vec{w} \in V$, where $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ and $\vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$. Use this setup to show that T is one-to-one; i.e., show that $\vec{v} = \vec{w}$.

Onto $\mathbb{R}^n$: Let $\vec{w} \in \mathbb{R}^n$, i.e., $\vec{w} = (c_1, c_2, \ldots, c_n)$ for some $c_i \in \mathbb{R}$. Use the definition of T to produce a vector in $\mathbb{V}$ that T maps to $\vec{w}$.

Therefore, V and $\mathbb{R}^n$ are isomorphic. □

So, in particular, every vector space of dimension n is isomorphic to $\mathbb{R}^n$. This means that all the things that we know about $\mathbb{R}^n$ as a vector space can be translated into results about any other n-dimensional vector space via a vector space isomorphism.

Part II (prepare for class Friday, April 12)

1. Consider $V = M_{3\times 3}$.

   (a) What is $\dim(M_{3\times 3})$? And thus, what is n such that $M_{3\times 3}$ is isomorphic to $\mathbb{R}^n$?

   (b) Using the standard basis for $M_{3\times 3}$ and the transformation described in the proof above, to what element of that $\mathbb{R}^n$ does the matrix $\begin{bmatrix} 1 & 2 & 50 \\ 7 & -100 & -50 \\ -7 & \frac{11}{2} & 1000 \end{bmatrix}$ get mapped?

Part III: Homework (due Wednesday, April 10 at the beginning of class)

1. Let $T\colon P_1 \to \mathbb{R}$ be the transformation such that $T(\vec{p}) = \displaystyle\int_{-1}^{1} p(x)\,dx$. Is T one-to-one? Justify your answer.

2. Let $T\colon C^1[0,1] \to \mathbb{R}$ such that $T(\vec{f}) = f(0)+2f'(0)+3f'(1)$. Is T one-to-one? Justify your conclusion.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors
- span of a set of vectors
- linearly independent
- linearly dependent
- reduced row echelon form
- pivot
- homogeneous system
- free variable
- row equivalent
- consistent system
- inconsistent system
- trace of a matrix
- transpose of a matrix
- inverse of a matrix
- elementary matrix

- transformation
- domain
- codomain
- range
- vector space (I will not ever ask you to define this on a quiz—the definition is way too long—but you should make sure you know what makes something a vector space)
- subspace
- basis
- finite-dimensional vector space
- dimension
- coordinate vector
- column space of A
- row space of A
- null space of A
- rank
- nullity
- linear transformation
- kernel
- range
- isomorphism
- isomorphic vector spaces