Part I (due Friday, March 22 at the beginning of class)

As we've mentioned in class before, it is often helpful to be able to use different bases for the same space.

Change of Basis Problem: Suppose we have two bases, $\mathcal{B}$ and $\mathcal{B}'$ for a vector space V. If $\vec{w} \in V$ and we change the basis for V from $\mathcal{B}$ to $\mathcal{B}'$ (or vice versa), how are $[\vec{w}]_{\mathcal{B}}$ and $[\vec{w}]_{\mathcal{B}'}$ related? (The square brackets indicate column vectors for the coordinate vectors with respect to the particular basis, but it doesn't really matter—that's just for convenience in some of the calculations here; we'll also use parentheses for these vectors when we write them out as lists of numbers.)

Note: if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\vec{v})_S = (c_1, c_2, \ldots, c_n)$$

is the coordinate vector of $\vec{v}$ relative to S, then the mapping $\vec{v} \to (\vec{v})_S$ (called the *coordinate map*) creates a one-to-one correspondence between vectors in V and vectors in $\mathbb{R}^n$.

In Two Dimensions We'll start in the two-dimensional case. Suppose we have two bases for our vector space V, $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ and $\mathcal{B}' = \{\vec{u}_1', \vec{u}_2'\}$ and we want to have a method for writing vectors in V that are already in terms of $\mathcal{B}'$ in terms of $\mathcal{B}$ instead. First, we'll deal with just changing the basis vectors: we have to the write the vectors in $\mathcal{B}'$ as linear combinations of the vectors in $\mathcal{B}$, i.e,

$$\vec{u}_1' = a\vec{u}_1 + b\vec{u}_2$$
$$\vec{u}_2' = c\vec{u}_1 + d\vec{u}_2.$$

This gives us the coordinate vectors for the basis vectors in $\mathcal{B}'$ with respect to the basis $\mathcal{B}$: $[\vec{u_1}']_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $[\vec{u_2}']_{\mathcal{B}} = \begin{bmatrix} c \\ d \end{bmatrix}$.

Now let $\vec{v} \in V$ and suppose the coordinate vector with respect to $\mathcal{B}'$ for some vector $\vec{v}$ in V is

$$[\vec{v}]_{\mathcal{B}'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix},$$

i.e.,

$$\vec{v} = k_1\vec{u}_1' + k_2\vec{u}_2'.$$

To find the coordinate vector with respect to $\mathcal{B}$ for $\vec{v}$ when we already have the coordinate vector with respect to $\mathcal{B}'$ for $\vec{v}$, we write

$$\vec{v} = k_1\vec{u}_1' + k_2\vec{u}_2 = k_1(a\vec{u}_1 + b\vec{u}_2) + k_2(c\vec{u}_1 + d\vec{u}_2) = (k_1a + k_2c)\vec{u}_1 + (k_1b + k_2d)\vec{u}_2,$$

which gives us that the coordinate vector with respect to $\mathcal{B}$ for $\vec{v}$ is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} k_1a + k_2b \\ k_1c + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\vec{v}]_{\mathcal{B}'}.$$

So we get the coordinate vector with respect to $\mathcal{B}$ for $\vec{v}$ by multiplying the coordinate vector with respect to $\mathcal{B}'$ for $\vec{v}$ by a matrix P whose columns are the coordinates of the basis vectors in $\mathcal{B}'$ relative to the basis $\mathcal{B}$.

In n Dimensions Now let's generalize to n dimensions.

Theorem 1 (Solution to Change of Basis Problem)**.** *If $\vec{v} \in V$ and if $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\mathcal{B}' = \{\vec{u}'_1, \vec{u}'_2, \ldots, \vec{u}'_n\}$ are bases for V, then*

$$[\vec{v}]_{\mathcal{B}} = P[\vec{v}]_{\mathcal{B}'},$$

where the columns of P are the coordinate vectors of the basis vectors in $\mathcal{B}'$ relative to the basis $\mathcal{B}$:

$$[\vec{u}'_1]_{\mathcal{B}},\ [\vec{u}'_2]_{\mathcal{B}},\ \ldots\ [\vec{u}'_n]_{\mathcal{B}}.$$

Note: P is called the *transition matrix* from $\mathcal{B}'$ to $\mathcal{B}$; we write $P_{\mathcal{B}'\to\mathcal{B}}$ to show the bases. We can also go the other way, in which case we have

$$P_{\mathcal{B}\to\mathcal{B}'} = [[\vec{u}_1]_{\mathcal{B}'} \mid [\vec{u}_2]_{\mathcal{B}'} \mid \cdots \mid [\vec{u}_n]_{\mathcal{B}'}]$$

Example 1.

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathcal{B}' = \left\{ \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \right\}$$

are bases for $\mathbb{R}^3$.

1. Find $P_{\mathcal{B}'\to\mathcal{B}}$.

   Solution: Call the vectors in $\mathcal{B}$ $\vec{v}_i$ and the vectors in $\mathcal{B}'$ $\vec{w}_i$. Then $P_{\mathcal{B}'\to\mathcal{B}} = [[\vec{w}_1]_{\mathcal{B}}\, [\vec{w}_2]_{\mathcal{B}}\, [\vec{w}_3]_{\mathcal{B}}]$, i.e., we need to write each $\vec{w}_i$ as a linear combination of the $\vec{v}_i$'s. So we have

   $$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{w}_1,$$

   which is

   $$\begin{bmatrix} 2 & 1 & 1 & 6 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   giving us the first column of the transition matrix being $(2, 1, 1)$. We can find the other columns similarly, and we then get

   $$P_{\mathcal{B}'\to\mathcal{B}} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

2. Express $\vec{v} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$ in terms of $\mathcal{B}'$ and then in terms of $\mathcal{B}$.

   Solution: To express this vector in terms of $\mathcal{B}'$, we solve the system

   $$\begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix} = a \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix},$$

   which gives us

   $$(\vec{v})_{\mathcal{B}'} = (1, 2, -2).$$

   To determine $(\vec{v})_{\mathcal{B}}$, we take $P_{\mathcal{B}'\to\mathcal{B}}[\vec{v}]_{\mathcal{B}'}$ to get

   $$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

3. Find $P_{\mathcal{B}\to\mathcal{B}'}$.

   We can do this similarly to before; we end up with

$$P_{\mathcal{B}\to\mathcal{B}'} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}$$

Note: $P_{\mathcal{B}\to\mathcal{B}'} \cdot P_{\mathcal{B}'\to\mathcal{B}} = I$.

Theorem 2. *If P is the transition matrix from a basis $\mathcal{B}'$ to a basis $\mathcal{B}$ for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.*

Idea of Proof.

$$(P_{\mathcal{B}'\to\mathcal{B}})\,(P_{\mathcal{B}\to\mathcal{B}'}) = (P_{\mathcal{B}\to\mathcal{B}}) \tag{1}$$

(since we apply matrices from right to left when we're multiplying them by a coordinate vector).

Maps a vector with respect to $\mathcal{B}$ to itself, so we conclude that

$$(P_{\mathcal{B}\to\mathcal{B}}) = I$$

Multiplying the first two matrices in (1) in the opposite order, we similarly get

$$(P_{\mathcal{B}\to\mathcal{B}'})\,(P_{\mathcal{B}'\to\mathcal{B}}) = (P_{\mathcal{B}'\to\mathcal{B}'}) = I,$$

so we conclude that these two matrices are inverses of each other. □

Reasonable: If $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are bases for a vector space V, then

$$P_{\mathcal{B}_2\to\mathcal{B}_3} P_{\mathcal{B}_1\to\mathcal{B}_2} = P_{\mathcal{B}_1\to\mathcal{B}_3},$$

where the first matrix on the left maps $\mathcal{B}_2$ coordinates to $\mathcal{B}_3$ coordinates, etc.

Computing Transition Matrices in $\mathbb{R}^n$

To find these transition matrices, we're really just solving a system of equations for each column of the transition matrix, with the same set of columns on the left every time. Thus, we can solve them all at once, like we did when we were finding inverse matrices.

To find the transition matrix $P_{\mathcal{B}\to\mathcal{B}'}$,

1. Form the matrix $[\mathcal{B}'|\mathcal{B}]$.
2. Row reduce.
3. Resulting matrix is $[I|P_{\mathcal{B}\to\mathcal{B}'}]$.
4. Pull out the transition matrix.

Part II

That was a long reading, so no Part II for Friday. We'll practice some change of basis in class Friday.

Part III: Homework (due Wednesday, March 27 at the beginning of class)

1. Suppose that A and B are $n \times n$ matrices and A is invertible. What can you say about the row space of AB and the row space of B? Prove your answer.
2. True or false: There exists an invertible matrix A and a noninvertible matrix B with the same row spaces. If true, prove; if false, give an explained counterexample.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors
- span of a set of vectors
- linearly independent
- linearly dependent
- reduced row echelon form
- pivot
- homogeneous system
- free variable
- row equivalent
- consistent system
- inconsistent system
- trace of a matrix
- transpose of a matrix
- inverse of a matrix
- elementary matrix
- transformation
- domain

- codomain
- range
- vector space (I will not ever ask you to define this on a quiz—the definition is way too long—but you should make sure you know what makes something a vector space)
- subspace
- basis
- finite-dimensional vector space
- dimension
- coordinate vector
- column space of A
- row space of A
- null space of A
- rank
- nullity