Part I (due Friday, March 15 at the beginning of class)

Definition 1. *Let A be an $m \times n$ matrix.*

The column space of A, *denoted $C(A)$, is the set of all linear combinations of the columns of A (i.e., is the span of the columns of A).*

The row space of A, *denoted $C(A^T)$, is the set of all linear combinations of the rows of A.*

The null space of A *is the solution space of the system $A\vec{x} = \vec{0}$.*

The row space, column space, and null space of an $m \times n$ matrix A are sometimes called the *fundamental matrix spaces.* We study them because they tell us useful information about the matrix and (surprise!) they also give us a new perspective on many of the questions we've already been asking this semester. As we study them, we're looking for both how these spaces are related to each other and how they're related to solutions to the system of linear equations $A\vec{x} = \vec{b}$.

As we've talked about many times this semester, we can think of the matrix multiplication $A\vec{x}$ as a linear combination of the columns of A with coefficients coming from the vector $\vec{x}$. One of the questions we've asked in relation to this is what vectors $\vec{b}$ we can put on the right hand side of the equation $A\vec{x} = \vec{b}$ to get a system that has a solution (a system for which there is a vector $\vec{x}$ that we can multiply by A to get the vector $\vec{b}$). Looking at that question through the lens of $A\vec{x}$ being a linear combination of the columns of A, we're asking what vectors $\vec{b}$ can be written as a linear combination of the columns of A, which is the same as asking what vectors $\vec{b}$ are in span of the columns of A, which in turn is the same as asking what vectors $\vec{b}$ are in the column space $C(A)$. This reasoning gives us the following proposition.

Proposition 1. *A linear system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b} \in C(A)$.*

One of the things we'd like to be able to do when we have a subspace of a vector space is to give a basis for that subspace—as we've seen, this allows us to establish a coordinate system for the subspace, which will be useful. Toward that end, we consider the following result. Recall that a matrix is in row echelon form when all rows of zeros are at the bottom, the first nonzero entry in every row is a 1, and each leading one is to the right of the leading one in the row above it.

Proposition 2. *The nonzero rows of a matrix R in row echelon form create a basis for the row space of R.*

Proof. Let R be a matrix in row echelon form, so any zero rows are at the bottom of the matrix. Let $\vec{r}_j$ denote the jth non-zero row of R. We want to show that the set of vectors $\mathcal{B} = \{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k\}$, consisting of all nonzero rows of R, form a basis for the row space of R. Thus, we must show that $\mathcal{B}$ is linearly independent and spans $C(R^T)$.

Since $C(R^T)$ is defined to be the set of all linear combinations of the rows of R and zero vectors add nothing the span of a set of vectors, $\mathcal{B}$ must span $C(R^T)$.

We'll show that $\mathcal{B}$ is linearly independent by contradiction. So assume that $\mathcal{B}$ is linearly dependent. Then one of the vectors in $\mathcal{B}$, call it $\vec{r}_p$, can be written as a linear combination of the other vectors in $\mathcal{B}$:

$$\vec{r}_p = c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_{p-1}\vec{r}_{p-1} + c_{p+1}\vec{r}_{p+1} + \cdots + c_k\vec{r}_k.$$

Subtracting the expression on the left from both sides, we get

$$\vec{r}_p - (c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_{p-1}\vec{r}_{p-1} + c_{p+1}\vec{r}_{p+1} + \cdots + c_k\vec{r}_k) = \vec{0}.$$

This equation can be thought of as performing a series of elementary row operations on R since we're just adding a scalar multiple of another row to $\vec{r}_p$ for each term we're subtracting. Thus, we can, through a series of elementary row operations on R, turn row p into the zero vector. But this is a contradiction since then R would not be in row echelon form with all the zero rows at the bottom. Hence, $\mathcal{B}$ must be a linearly independent set.

Therefore, $\mathcal{B}$ forms a basis for $C(R^T)$. $\square$

And one more useful result:

Proposition 3. *Elementary row operations do not change the row space or the null space of a matrix.*

Combining the last two results, we get that the nonzero rows of rref(A) form a basis for the row space of A. In other words, to find a basis for the row space of A, all we have to do is row reduce A, pick out all the rows of rref(A) that have leading one's in them, and put them in a set—that set of vectors is a basis for the row space of A.

Reading Question(s)

1. Find a basis for the row space of $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 8 & 10 & 2 & 0 \\ 7 & 8 & 2 & -3 \end{bmatrix}$.

Part II (due Wednesday, March 13)

1. Given an $m \times n$ matrix A, prove that $N(A)$ is a subspace of $\mathbb{R}^n$.
2. Find a basis for the span of the following sets of vectors (hint: use the last paragraph of the reading to help you).
   (a) $\{(1,5),(7,2)\}$
   (b) $\{(-1,-3),(4,12)\}$
   (c) $\{(3,6,5),(2,1,2),(12,15,15)\}$
   (d) $\{(-1,1,-1,1),(-5,-9,-7,-1),(0,7,1,3),(-6,-1,-7,3),(2,5,3,1)\}$
3. Describe the column space of $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$.

Part III: Homework (due Friday, March 15 at the beginning of class)

1. True or false (If true, prove; if false, give a specific, explained counterexample):
   (a) Every linearly independent set of five vectors in $\mathbb{R}^5$ is a basis for $\mathbb{R}^5$.

(b) Every set of five vectors that spans $\mathbb{R}^5$ is a basis for $\mathbb{R}^5$.

(c) Every set of vectors that spans $\mathbb{R}^n$ contains a basis for $\mathbb{R}^n$.

(d) Every linearly independent set of vectors in $\mathbb{R}^n$ is contained in some basis for $\mathbb{R}^n$.

(e) If S is a set of 2×2 matrices, then S is linearly independent if and only if the set of first columns of matrices in S and the set of second columns of the matrices in S are both linearly independent.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors
- span of a set of vectors
- linearly independent
- linearly dependent
- reduced row echelon form
- pivot
- homogeneous system
- free variable
- row equivalent
- consistent system
- inconsistent system
- trace of a matrix
- transpose of a matrix
- inverse of a matrix
- elementary matrix
- transformation
- domain
- codomain
- range
- vector space (I will not ever ask you to define this on a quiz—the definition is way too long—but you should make sure you know what makes something a vector space)
- subspace

- basis
- finite-dimensional vector space
- dimension
- coordinate vector