Part I (due Wednesday, March 6 at the beginning of class)

Recall the definition of linear independence (note that it works in any vector space, not just $\mathbb{R}^n$):

Definition 1. *If $S = \vec{v}_1, \vec{v_2}, \ldots, \vec{v}_r$ is a nonempty set of vectors, then the vector equation*

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0} \tag{1}$$

has at least one solution:

$$k_1 = 0,\ k_2 = 0,\ \ldots,\ k_r = 0.$$

If the trivial solution is the only solution to (1)*, then S is called a* linearly independent set*. If there are other solutions to* (1)*, then S is a* linearly dependent set*.*

Here are some examples; the first few are review with particular $\mathbb{R}^n$'s.

Example 1.

- In $\mathbb{R}^2$
  - Is $S = \{(2,4), (-4,-8)\}$ linearly independent?
    We consider
    $$k_1(2,4) + k_2(-4,-8) = \vec{0},$$
    which gives the homogeneous system of equations
    $$\begin{aligned} 2k_1 - 4k_2 &= 0 \\ 4k_2 - 8k_2 &= 0 \end{aligned}$$
    Since the rref of the coefficient matrix is not the identity matrix, this system has nontrivial solutions, so the set of vectors is linearly dependent.
  - Is $S = \{(2,3), (-1,2)\}$ linearly independent?
    We consider
    $$k_1(2,3) + k_2(-1,-2) = \vec{0},$$
    which gives the homogeneous system of equations
    $$\begin{aligned} 2k_1 - k_2 &= 0 \\ 3k_2 - 2k_2 &= 0 \end{aligned}$$
    Since the rref of the coefficient matrix is I, this system has only the trivial solution, so the set of vectors is linearly independent.
- In $\mathbb{R}^3$, is $S = \{(1,2,3), (4,1,0), (-1,2,1)\}$ linearly independent?
  We consider
  $$k_1(1,2,3) + k_2(4,1,0) + k_3(-1,2,1) = \vec{0},$$
  which gives the homogeneous system of equations
  $$\begin{aligned} k_1 + 4k_2 - k_3 &= 0 \\ 2k_1 + k_2 + 2k_3 &= 0 \\ 3k_1 + \quad + k_3 &= 0 \end{aligned}$$
  Since the rref of the coefficient matrix is I, this system has only the trivial solutions, so the set of vectors is linearly independent.

- In P_3, is $S = \{x, 5, 3x^3 + 4x^2 + x, 9x^3 + 12x^2 - 4x + 1\}$ linearly independent?

  We consider
  $$k_1(x) + k_2(5) + k_3(3x^3 + 4x^2 + x) + k_4(9x^3 + 12x^2 - 4x + 1) = \vec{0},$$
  which is equivalent to
  $$(5k_2 + k_4) + (k_1 + k_3 - 4k_4)x + (4k_3 + 12k_4)x^2 + (3k_3 + 9k_4)x^3 = 0.$$
  Since this equation must be true for all $x \in \mathbb{R}$, all the coefficients must be equal to zero, which gives the homogeneous system of equations
  $$\begin{aligned} 5k_2 + \quad k_4 &= 0 \\ k_1 + \quad k_3 - 4k_4 &= 0 \\ 4k_3 + 12k_4 &= 0 \\ 3k_3 + 9k_4 &= 0 \end{aligned}$$
  Since the rref of the coefficient matrix is not I, this system has nontrivial solutions, so the set of vectors is linearly dependent.

We've talked about bases for before in the context of $\mathbb{R}^n$, but we haven't done much with them, so we're going to revisit them now and extend the idea to bases for general vector spaces.

Standard Basis Vectors

$\mathbb{R}^2$: $(1, 0)$ and $(0, 1)$

$\mathbb{R}^3$: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

$\mathbb{R}^n$: $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, …, $(0, \ldots, 0, 1)$

- These are unit vectors along the axes for which every vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors. **Reading Question 1:** How do we know that every vector can be written as a linear combination of the standard basis vectors?
- They're linearly independent. **Reading Question 2:** How do we know this?
- They give us a coordinate system (think of coordinates in the Cartesian plane).

Our goal is to extend this idea: these are not the only linearly independent sets of vectors that span $\mathbb{R}^n$. We've seen this definition before:

Definition 2. *If V is any vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a set of vectors in V, then S is called a* basis *for V if S is linearly independent and* $\text{span}(S) = V$.

Note: a basis is the vector space generalization of a coordinate system in $\mathbb{R}^n$.

Example 2. Is $S = \{(2, 3), (-1, 2)\}$ a basis for $\mathbb{R}^2$?

- linearly independent? $c_1(2,3)+c_2(-1,2)=(0,0)$ gives the matrix equation

$$A\vec{c}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

  and $\text{rref}(A)=I$, so we have linear independence.

- spans $\mathbb{R}^2$? $c_1(2,3)+c_2(-1,2)=(b_1,b_2)$ gives the matrix equation

$$A\vec{c}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix},$$

  and $\text{rref}(A)=I$, so this equation has a solution for any $\vec{b}$, hence the set spans $\mathbb{R}^2$.

Reading Question 3: How could we answer the linear independence and span question at the same time?

Part II (prepare for Wednesday, March 6)

1. Are the following sets linearly independent or linearly dependent in the given vector spaces?
   (a) $S=\{1, x^2-2x, 5(x-1)^2\}$ in P_2
   (b) $S=\left\{\begin{bmatrix}4 & 2\\ 1 & 0\end{bmatrix}, \begin{bmatrix}3 & 1\\ 1 & 4\end{bmatrix}, \begin{bmatrix}8 & -13\\ 1 & 44\end{bmatrix}\right\}$ in M_2, the space of 2×2 matrices with real number entries
   (c) $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ in any vector space V

Part III: Homework (due Friday, March 15 at the beginning of class)

1. True or False (if true, prove; if false, give an explained counterexample):
   (a) There is a vector space consisting of exactly two distinct vectors.
   (b) The set of all upper triangular $n\times n$ matrices (all entries below the main diagonal are 0; the main diagonal and above can be anything) is a subspace of the vector space of all $n\times n$ matrices.
   (c) The polynomials $x-1$, $(x-1)^2$, and $(x-1)^3$ span P_3, the set of all polynomials of degree three or less with real coefficients.

Running list of vocabulary words that could be a quiz word

- linear equation
- system of linear equations
- linear combination of a set of vectors
- span of a set of vectors
- linearly independent

- linearly dependent
- reduced row echelon form
- pivot
- homogeneous system
- free variable
- row equivalent
- consistent system
- inconsistent system
- trace of a matrix
- transpose of a matrix
- inverse of a matrix
- elementary matrix
- transformation
- domain
- codomain
- range
- vector space (I will not ever ask you to define this on a quiz—the definition is way too long—but you should make sure you know what makes something a vector space)
- subspace