Part I: Reading (due at the beginning of class Monday, April 15)

Optional: Ratio Test Proof

Let's see how we can prove the Ratio Test (for those who are interested). Suppose we have a series $\sum a_n$ such that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

We'll first consider the case where $L < 1$. Since $L < 1$, there is space between L and 1, so we're going to choose a number r that's in that space, i.e., we choose r such that $L < r < 1$. We have that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \text{ and } L < r,$$

which implies that the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ is eventually less than r. A more rigorous way to say that is that there is a natural number N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r$$

for all values of $n \geq N$.

So, in the case that $n \geq N$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| < r,$$

which we can rearrange as

$$|a_{n+1}| < |a_n|r.$$

Since this inequality holds for all values of $n \geq N$, we have, when $n = N$,

$$|a_{N+1}| < |a_N|r. \tag{1}$$

When $n = N + 1$, substituting the result from (1), we have

$$|a_{N+2}| < |a_{N+1}|r < (|a_N|r)r = |a_N|r^2. \tag{2}$$

Repeating this process with $n = N + 2$ and substituting the result from (2), we have

$$|a_{N+3}| < |a_{N+2}|r < (|a_N|r^2)r = |a_N|r^3.$$

In general, for $n = N + k$, we will have

$$|a_{N+k}| < |a_{N+(k-1)}|r < (|a_N|r^{k-1})r = |a_N|r^k. \tag{3}$$

Now consider the series

$$\sum_{k=1}^{\infty}|a_N|r^k = |a_N|r + |a_N|r^2 + |a_N|r^3 + \cdots. \tag{4}$$

This is a geometric series with $r < 1$ (remember, we choose r above so that it was less than 1), so it's a convergent geometric series. By (4), we know that

$$|a_n| < |a_N|r^k$$

for all values of $n \geq N$, so by the Comparison Test, the series $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, $\sum_{n=1}^{\infty} a_n$ converges absolutely, and we've proven the Ratio Test in the case that $L < 1$.

Now suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1.$$

Then the ratio $\left| \frac{a_{n+1}}{a_n} \right| > 1$ eventually; i.e., there is an N such that for all $n \geq N$, $\left| \frac{a_{n+1}}{a_n} \right| > 1$. Rearranging as in the previous case, we get $|a_{n+1}| > |a_n|$ for all $n \geq N$, which means that we can't have $a_n \to 0$. Thus, by the nth term test for divergence, we conclude that $\sum a_n$ diverges.

Not optional: Really brief introduction for our next topic

For your Part I reading questions, answer the questions below. We've been looking at infinite series, which are infinite sums of real numbers. Now we're going to expand our idea of what we're allowed to sum and include variables in our options (though we're still going to stick with real-valued variables). So, for example, we could consider the series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

We can regard this as a geometric series.

1. What is r?
2. When will this series converge?
3. When the series does converge, to what value does it converge?

Part II: Exercises (prepare for class Monday, April 15)

Spend a focused half hour on the Determining Convergence handout. We'll present some of the problems from it on Monday.

Part III: Homework Problems (due Wednesday, April 17 at the beginning of class)

1. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n!}$. Find a value of n such that s_n, the nth partial sum, approximates the series with an error of at most 10^{-5}. Then compute this approximation. (Note: feel free to use Wolfram Alpha or other computational devices as needed, but explain what you're doing.)

2. Find all the positive integers k for which the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

converges. Hint for this one and the next one: the Ratio Test is your friend.

3. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

converge? For those values of x, what, if anything, does this result allow you to say about $\lim_{n\to\infty} \frac{x^n}{n!}$?

Reminder: Celebration of Learning #3 on Friday, April 12

You may bring a hand-written notecard with you to the Celebration of Learning (you can use both sides or bring two one-sided cards).