Part I (due at the beginning of class Monday, September 15)

If we know that we have continuous functions, we can put them together in various ways to get more continuous functions. This theorem gives us the basic options for doing so:

Theorem 1 (Algebraic Continuity Theorem)**.** *If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following functions are also continuous at $x = c$:*

(i) $f(x) + g(x)$

(ii) $kf(x)$ *for any constant* k

(iii) $f(x)g(x)$

(iv) $\dfrac{f(x)}{g(x)}$ *as long as* $g(c) \neq 0$.

The fact that each of combinations of functions are continuous follows from the Basic Limit Laws; we'll look at details for (iii).

(iii). Suppose that $f(x)$ and $g(x)$ are continuous at $x = c$. To show that the product $f(x)g(x)$ is also continuous at $x = c$, we need to show that it satisfies the definition of continuous at a point. In other words, we need to show that $\lim_{x \to c} f(x)g(x) = f(c)g(c)$.

Since f and g are both continuous at $x = c$, we have that

$$\lim_{x \to c} f(x) = f(c) \text{ and } \lim_{x \to c} g(x) = g(c).$$

So by the Product Limit Law, we have

$$\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = f(c)g(c).$$

Thus, $f(x)g(x)$ is continuous at $x = c$. □

Classes of functions that are continuous

We can use the Algebraic Continuity Theorem to determine that many different classes of functions are continuous. Here are several:

- polynomials are continuous for all real numbers becayse polynomials are sums of functions of the form ax^m, all of which are continuous.
- If $p(x)$ and $q(x)$ are polynomials, then $\dfrac{p(x)}{q(x)}$ is continuous on its domain (for all values $x = c$ such that $q(c) \neq 0$).
- $f(x) = x^{\frac{1}{n}}$ is continuous on its domain for any natural number n (the natural numbers are $\{1, 2, 3, 4, \ldots\}$).

- $f(x) = \sin x$ and $f(x) = \cos x$ are continuous for all real numbers.
- $f(x) = b^x$ is continuous for all real numbers x as long as $b > 0$ and $b \neq 1$.
- $f(x) = \log_b x$ is continuous for all $x > 0$ as long as $b > 0$ and $b \neq 1$.
- If g is continuous at $x = c$ and f is continuous at $x = g(c)$, then the composite function $F(x) = f(g(x))$ is continuous at $x = c$.

As we mentioned at the end of class Friday, we like continuous functions when we're finding limits because for a continuous function, we get to just plug in the value that x is approaching in order to find the limit. However, there are a lot of functions that we can't do that for; you saw several on the Introduction to Limits handout where if you plugged in the value x was approaching, you got zero in the denominator, but when you estimated the limit numerically (using a table of values), you got an actual number.

Let's consider this example:

Example 1. Find $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x - 2}$.

If we plug in $x = 2$, we get zero in the denominator, so we can't do that and we can't get the limit by just taking the limit of the top and the bottom. When $x \neq 2$, though, we have

$$\frac{x^2 - 5x + 6}{x - 2} = \frac{(x-2)(x-3)}{x-2} = x - 3.$$

So the function we have coincides with the continuous function $x - 3$ everywhere except at $x = 2$. Try graphing the original function $\dfrac{x^2 - 5x + 6}{x - 2}$ and the function $g(x) = x - 3$ to see this visually.

Since $x \neq 2$ when we're taking the limit (we only care about what's happening at values close to 2, not actually equal to 2), we have

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2} = \lim_{x \to 2} (x - 3) = 1.$$

We say that $f(x)$ has an *indeterminate form* (or is *indeterminate*) at $x = c$ if substituting c for x gives an undefined expression of one of the following types:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad \text{and some others}$$

If we have a function like this for which we need to find the limit as $x \to c$, we can't just substitute c in directly, so we first transform the function algebraically, if we can, into a function that is continuous at $x = c$ so that we can then find the limit.

We'll look at several other ways to transform functions in class.

Reading Question(s)

1. Finish through question 3 on the Continuity handout. We'll discuss this at the beginning of class on Monday and I'll walk around to check that you've done it, but I will not collect the handout.

Recall, what you turn in for Part I should have 3 subparts, as mentioned in the syllabus:

(a) Your response(s) to the reading question(s).

(b) Your own questions/comments on the reading.

(c) The amount of time you spent on Part I (including the time spent reading/watchinh).

Part II: WeBWorK (due Saturday, September 13, by 11 PM)

Click here for your WeBWorK assignment. Complete the DW 6 WeBWorK assignment.

Part III: Homework Problems (due Wednesday, September 17 at the beginning of class)

Review the homework guidelines and the sample homework in the syllabus to ensure that the solutions you turn in meet the guidelines.

1. A parking lot charges $4 for the first hour or part thereof and $3 for each subsequent hour or part thereof up to a daily maximum of $15.

   (a) Write a function $f(x)$ to model this situation for someone parking in the lot, where x is the amount of time that person is parked in the lot. Hint: the ceiling function $\lceil x \rceil$ = least integer n such that $n \geq x$ is helpful here. For example, $\lceil 2.5 \rceil = 3$, $\lceil 7 \rceil = 7$, and $\lceil 2\pi \rceil = 7$.

   (b) Is your function from part (a) continuous or discontinuous? If it is discontinuous, discuss the significance of the discontinuities for someone who parks there.

   (c) Sketch a graph of this function.

   (d) The greatest integer function $[[x]]$, also called the floor function (and generally denoted $\lfloor x \rfloor$ in that case), returns the greatest integer less than or equal to x, e.g., $[[e]] = 2$ and $[[1]] = 1$. Give an example of a situation in which you would use some version of the greatest integer function to model the situation. Explain the significance of any discontinuities that arise in your example.